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Negative-dimensional integration revisited

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Abstract. Feynman diagrams are the best tool we have to study perturbative quantum field theory. For this very reason the development of any new technique that allows us to compute Feynman integrals is welcome. By the middle of the 1980s, Halliday and Ricotta suggested the possibility of using negative-dimensional integrals to tackle the problem. The aim of this work is to revisit the technique as such and check on its possibilities. For this purpose, we take a box diagram integral contributing to the photon–photon scattering amplitude in quantum electrodynamics using the negative-dimensional integration method. Our approach enables us to quickly reproduce the known results as well as six other solutions as yet unknown in the literature. These six new solutions arise quite naturally in the context of negative-dimensional integration method, revealing a promising technique to handle Feynman integrals.

1. Introduction

Scattering amplitudes, radiative corrections, β functions of renormalization group, etc, all require the computation of Feynman integrals [1], which are more complex to evaluate, the more loops in a given diagram one has. Since this approach is still the best technique we have to study quantum field theory (QFT) perturbatively, solving Feynman integrals becomes basic to any serious study of physical processes of interest involving those quantities. Such computations become harder to do not only with an increasing number of loops, but also with an increasing number of massive particles in the intermediate states.

The standard way to solve such integrals starts with the introduction of Feynman parameters, Wick rotation and then finally, integration. This method is somewhat tedious and sometimes it is not possible to exactly solve the parametric integrals. For this reason physicists developed several other techniques to calculate Feynman integrals [2]. A technique known as negative-dimensional integration method (NDIM) [3–5] has also been considered to tackle the problem.

One of the outstanding features of NDIM is that the complexities of performing D-dimensional momentum integrals are transferred to the easier task of solving systems of linear algebraic equations. NDIM has allowed us to recover very easily the two known hypergeometric series representations for the pertinent Feynman integral. Another outstanding feature of NDIM—and this is in our opinion its greatest potential—is that it simultaneously gives six new results for the integral in question in a very straightforward manner. Each of them is valid in certain regions of external momenta and are related to the others by analytic continuation, either directly or indirectly.

Hypergeometric functions of one and two variables have many well known analytic continuation formulae but as the number of variables increases—as far as we know—the

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fewer the known relations [6] are. On the other hand, since NDIM provides us with very many simultaneous results, which in principle must be connected by analytic continuation, we come to the realization that it is not only a very powerful technique to work out Feynman integrals but an elegant approach to check on analytic continuation properties of the resulting functions as well. Consider our present case: in all we have eight distinct solutions for the Feynman integral for the photon–photon scattering, two of which have already been obtained in the literature and six new ones, which are connected with each other by suitable analytic continuation formulae.

One could rightfully ask: Why do we want so many distinct results at the same time? We give some good arguments for this. First, if we have only one† or at most two‡ results in distinct regions of the external momenta, all the other regions must be worked out entirely through the analytic continuation formulae, which is not always an easy task to perform and is certainly very time-consuming. Secondly, the important special case of forward scattering in the relativistic regime cannot be dealt with if one has only the two known hypergeometric series representations for the scalar Feynman integral relative to the photon–photon scattering. These series are unsuitable for handling this special case because of the very nature of their variables. The same reasoning applies to the backward scattering. Thirdly, our results are expressed in a compact form that can be transformed—by an appropriate integral representation—into the more cumbersome standard form in terms of dilogarithms, if one wishes. Fourthly, we can identify the branch points and singularities of Feynman integrals directly from their hypergeometric series representations. Fifthly and lastly, since any two distinct solutions are related by analytic continuation, NDIM is an elegant and economical way of obtaining analytic continuation formulae among hypergeometric series (see section 5).

The outline of our paper is as follows. In section 2 we give a brief review of the methodology to be employed and compute the integral proper, writing down the two well known hypergeometric series representations for it in section 3. In section 4 we present the six new solutions of the given integral and in section 5 we discuss these new results. In section 6 we conclude.

2. Integration in negative dimensions

NDIM was introduced by Halliday and Ricotta [3] some years ago. Here we present a brief review of this technique. Basically one performs an analytic continuation

$$\int \frac{\mathrm{d}^D q}{(A)(B)(C)\ldots} \xrightarrow{AC} \int \mathrm{d}^D q\, (A)(B)(C)\ldots \tag{1}$$

so that one gets a polynomial integral in $D < 0$§ from a rather complicated one in $D > 0$. We then solve it in $D < 0$ and go back [4] to $D > 0$, through another analytic continuation. One of the advantages of NDIM is that simultaneously we get several hypergeometric series representations for the integral in $D > 0$, i.e. we obtain expressions for all the possible regions of the external momenta.

† Via parametric integration [1].

‡ Via Mellin-Barnes' integration [2, 7].

§ The negative-dimensional linear operator object here can be seen from the viewpoint of positive-dimensional fermionic integration [8].

We start from the relation [3–5] between a Gaussian integral and its counterpart in negative dimensions

$$\int \mathrm{d}^D q \ \exp(-\lambda q^2) = \left(\frac{\pi}{\lambda}\right)^{D/2} = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda^j}{j!} \int \mathrm{d}^D q \ (q^2)^j \tag{2}$$

where in the last step we have expanded the exponential function in Taylor series. Just as in dimensional regularization [9] we take this expression as the definition of the negative D-dimensional integral [10]. The middle term is an analytic function of D so the integral on the right-hand side is also an analytic function of D [11, 12].

From this equation we get,

$$\int \mathrm{d}^D q \ (q^2)^j = (-1)^j \pi^{D/2} \delta_{D/2,-j} \Gamma(1+j). \tag{3}$$

In a similar way, we can solve, e.g.

$$J(i,j,k,l;m) = \int \mathrm{d}^D q \ (q^2 - m^2)^i [(q-p)^2 - m^2]^j [(q-k_1)^2 - m^2]^k [(q-k_2)^2 - m^2]^l \tag{4}$$

whose counterpart in $D > 0$ is the integral

$$K(i,j,k,l;m) = \int \frac{\mathrm{d}^D q}{(q^2 - m^2)^i [(q-p)^2 - m^2]^j [(q-k_1)^2 - m^2]^k} \frac{1}{[(q-k_2)^2 - m^2]^l}. \tag{5}$$

This is one of the integrals that contributes to the photon–photon scattering amplitude in QED and it is the one we want to evaluate in our 'lab test' for NDIM. Of course, since the external photons are real particles, they are on-shell, i.e. we consider here that $k_1^2 = k_2^2 = (p-k_1)^2 = (p-k_2)^2 = 0$ (see figure 1).

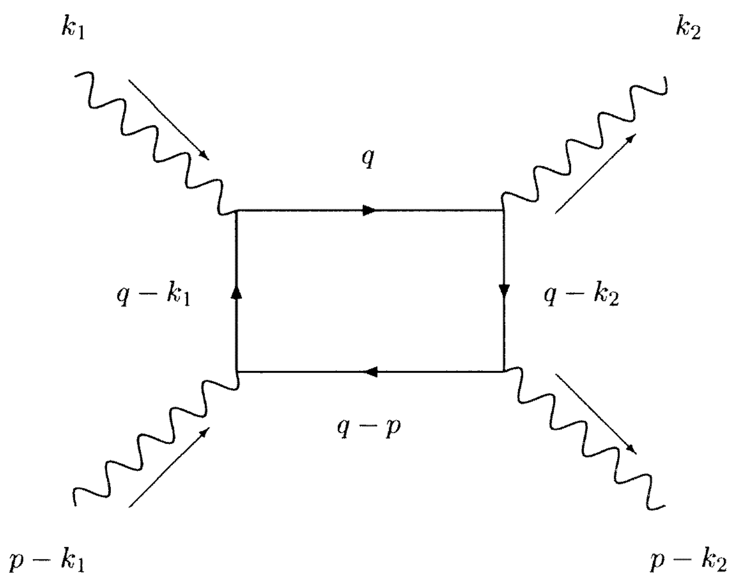

Figure 1. Feynman diagram for photon–photon scattering in the s-channel.

So, to begin with, let our ‘Gaussian integral’ be

$$I = \int \mathrm{d}^D q \ \exp\{-\alpha(q^2 - m^2) - \beta[(q-p)^2 - m^2] - \gamma[(q-k_1)^2 - m^2] \\ -\omega[(q-k_2)^2 - m^2]\}. \tag{6}$$

Completing the square, integrating over q and expanding the exponential, we get

$$I = \pi^{D/2} \sum_{n_i=0}^{\infty} \frac{(-s)^{n_1}(-t)^{n_2}(-m^2)^{n_3}\alpha^{n_2+n_4}\beta^{n_2+n_5}\gamma^{n_1+n_6}\omega^{n_1+n_7}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!}\Gamma\left(1+n_3-n_1-n_2-\frac{D}{2}\right) \tag{7}$$

where s and t are the Mandelstam variables (see figure 1) and m here is the mass of the virtual matter fields. Since we use a multinomial expansion the sum index above must satisfy the constraint

$$n_4 + n_5 + n_6 + n_7 = n_3 - n_1 - n_2 - \frac{D}{2}.$$

On the other hand, expanding the exponential of (6), we have

$$I = \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l}\alpha^i\beta^j\gamma^k\omega^l}{\Gamma(1+i)\Gamma(1+j)\Gamma(1+k)\Gamma(1+l)} J(i,j,k,l;m) \tag{8}$$

where we define our negative-dimensional integral, equation (4). Comparing the expressions (7) and (8) we obtain a general relation for the integral $J(i,j,k,l;m)$,

$$J(i,j,k,l;m) = \pi^{D/2}\Gamma(1+i)\Gamma(1+j)\Gamma(1+k)\Gamma(1+l) \\ \times \sum_{n_i=0}^{\infty} \frac{(-s)^{n_1}(-t)^{n_2}}{n_1!n_2!n_3!n_4!} \frac{(-m^2)^{n_3}}{n_5!n_6!n_7!} \delta_{n_2+n_4,i}\delta_{n_2+n_5,j}\delta_{n_1+n_6,k}\delta_{n_1+n_7,l} \\ \times\Gamma\left(1+n_3-n_1-n_2-\frac{D}{2}\right). \tag{9}$$

The several Kronecker delta’s lead to a system of linear algebraic equations linking the sum indices n_i, with five equations and seven variables, i.e.

$$\begin{aligned} n_2 + n_4 &= i \\ n_2 + n_5 &= j \\ n_1 + n_6 &= k \\ n_1 + n_7 &= l \\ n_4 + n_5 + n_6 + n_7 &= n_3 - n_1 - n_2 - \tfrac{1}{2}D. \end{aligned} \tag{10}$$

Since there are fewer equations than the total number of indices, the system above can in fact be solved if and only if we leave two free indices: that is, the result will be given as a double series. There are many ways we can choose these two free indices: indeed altogether there are 21 distinct ways. Of these, six will lead to unsolvable systems, i.e. trivial solutions, which are discarded. Of the remaining 15 which are non-trivial solutions we define eight sets, according to their variables, and each set is a basis generating the corresponding space of functions.

3. Hypergeometric series representations

The result we obtained in the previous section, equation (9), is written in the negative D region (and positive exponents of propagators). We must bring this result back to our real physical world, that of positive D and negative exponents of propagators. The technique that allows us to carry out the analytic continuation to the positive D region is explained in detail in [4].

Solving the system we find five hypergeometric series which can be divided into two sets according to its variables, $\{I_1\}$, and $\{I_2, I_3, I_4, I_5\}$. The solution in the first category is

$$I_1 = \left(\frac{\pi}{2}\right)^{D/2} \frac{2\sqrt{\pi}(-2m^2)^{\sigma}\Gamma(-\sigma)}{\Gamma(\frac{1}{2}-\frac{\sigma}{2}+\frac{D}{4})\Gamma(-\frac{\sigma}{2}+\frac{D}{4})} \sum_{n_1,n_2=0}^{\infty} \left(\frac{s}{4m^2}\right)^{n_1} \times \left(\frac{t}{4m^2}\right)^{n_2} \frac{(-i|n_1)(-j|n_1)(-k|n_2)(-l|n_2)(-\sigma|n_1+n_2)}{n_1!n_2!(-\frac{\sigma}{2}+\frac{D}{4}|n_1+n_2)(\frac{1}{2}-\frac{\sigma}{2}+\frac{D}{4}|n_1+n_2)} \tag{11}$$

where we define $\sigma = i+j+k+l+\frac{D}{2}$ and use the Pochhammer symbol

$$(a|k) \equiv (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Substituting $i=j=k=l=-1$, we get the integral (5) with exponents corresponding to the one we want to calculate for box diagrams. Then, the first hypergeometric series representation yields (in four dimensions)

$$J_1(-1,-1,-1,-1;m) = \frac{\pi^2}{6m^4} F_3\left(1,1,1,1;\frac{5}{2}\middle|\frac{s}{4m^2},\frac{t}{4m^2}\right). \tag{12}$$

This result is exactly that obtained by Davydychev [7] using the Mellin–Barnes' representation for massive propagators [2]. Note that since we are in Euclidean space there is an overall factor i difference when compared with Davydychev's result, obtained in Minkowski space (his result has the extra factor i). This expression is symmetric in s and t and is non-vanishing for $s=t=0$. It allows us to read off the particular cases where the integral has three and two propagators respectively, and also the analytic continuation to other regions of external momenta. This expression is valid in the region of convergence of the series which defines the F_3 hypergeometric function [13, 6],

$$F_3(\alpha,\alpha',\beta,\beta';\gamma|x,y) = \sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!k!} \frac{(\alpha|j)(\alpha'|k)(\beta|j)(\beta'|k)}{(\gamma|j+k)} \tag{13}$$

where $|x|<1$ and $|y|<1$. In other words, it is valid below the threshold of pair production. It is also suitable for studying the non-relativistic limit since the Mandelstam variable s must be less than $4m^2$. We note that $s=4m^2$ defines the point where the process changes its nature, that is, there exists the possibility of pair creation and this fact manifests itself in the amplitude as a branch point in the Feynman integral [1, 12].

The hypergeometric function in (12) can be expressed in terms of its double integral representation [13], and from there one can arrive at the standard result expressed in terms of a rather cumbersome sum of logarithms and dilogarithms [7, 14].

The next set of solutions, also obtained by Davydychev [7], has variables $4m^2/s$ and $4m^2/t$, the inverse of the ones in the first solution. In the following we write down these four solutions:

$$I_2 = \frac{2\pi^{D/2}(-t)^l(-s)^j(-m^2)^{D/2+i+k}(-i|j)(-k|l)}{(1+i-j+k-l|\frac{D}{2}+j+l)} \sum_{n_1,n_2=0}^{\infty} \frac{1}{n_1!n_2!}\left(\frac{4m^2}{s}\right)^{n_1}$$

$$\times\left(\frac{4m^2}{t}\right)^{n_2}\frac{(-j|n_1)(-l|n_2)(\frac{1+i-j+k-l}{2}|n_1+n_2)}{(1+i-j|n_1)(1+k-l|n_2)(1+i+k+\frac{D}{2}|n_1+n_2)}$$
$$\times(1+\tfrac{1}{2}(i-j+k-l)|n_1+n_2) \tag{14}$$

$$I_3=\frac{2\pi^{D/2}(-t)^k(-s)^j(-m^2)^{\frac{D}{2}+i+l}(-i|j)(-l|k)}{(1+i+j-k-l|\frac{D}{2}+j+k)}\sum_{n_1,n_2=0}^{\infty}\frac{1}{n_1!n_2!}\left(\frac{4m^2}{s}\right)^{n_1}$$
$$\times\left(\frac{4m^2}{t}\right)^{n_2}\frac{(-j|n_1)(-k|n_2)(\frac{1+i-j-k+l}{2}|n_1+n_2)}{(1+i-j|n_1)(1-k+l|n_2)(1+i+l+\frac{D}{2}|n_1+n_2)}$$
$$\times(1+\tfrac{1}{2}(i-j-k+l)|n_1+n_2) \tag{15}$$

$$I_4=\frac{2\pi^{D/2}(-t)^k(-s)^i(-m^2)^{\frac{D}{2}+j+l}(-j|i)(-l|k)}{(1-i+j-k+l|\frac{D}{2}+i+k)}\sum_{n_1,n_2=0}^{\infty}\frac{1}{n_1!n_2!}\left(\frac{4m^2}{s}\right)^{n_1}$$
$$\times\left(\frac{4m^2}{t}\right)^{n_2}\frac{(-i|n_1)(-k|n_2)(\frac{1-i+j-k+l}{2}|n_1+n_2)}{(1-i+j|n_1)(1-k+l|n_2)(1+j+l+\frac{D}{2}|n_1+n_2)}$$
$$\times(1+\tfrac{1}{2}(-i+j-k+l)|n_1+n_2) \tag{16}$$

and

$$I_5=\frac{2\pi^{D/2}(-t)^l(-s)^i(-m^2)^{\frac{D}{2}+j+k}(-j|i)(-k|l)}{(1-i+j+k-l|\frac{D}{2}+i+l)}\sum_{n_1,n_2=0}^{\infty}\frac{1}{n_1!n_2!}\left(\frac{4m^2}{s}\right)^{n_1}$$
$$\times\left(\frac{4m^2}{t}\right)^{n_2}\frac{(-i|n_1)(-l|n_2)(\frac{1-i+j+k-l}{2}|n_1+n_2)}{(1-i+j|n_1)(1+k-l|n_2)(1+j+k+\frac{1}{2}D|n_1+n_2)}$$
$$\times(1+\tfrac{1}{2}(-i+j+k-l)|n_1+n_2). \tag{17}$$

From these we construct the second hypergeometric series representation for the Feynman integral as just the linear combination,

$$J_2(i,j,k,l;m)=I_2+I_3+I_4+I_5. \tag{18}$$

Note that J_2 is the analytic continuation of J_1, see for example Erdélyi *et al* [13]. While J_1 is valid in the non-relativistic case and J_2 in the relativistic one, they do not cover all the possible regions of external momenta.

An important point here that one must be aware of is that even though singularities might appear in isolated terms of the RHS, for the special case when $i=j=k=l=-1$, the above equation (18) cannot be singular since the corresponding analytic continuation formula (12) is not.

To overcome this difficulty let us introduce small corrections in the parameters β and β' of the hypergeometric function $F_3(\alpha,\alpha',\beta,\beta';\gamma|x,y)$ [15]. This recourse corresponds to correcting the exponents of propagators [16]. In our case we take

$$\beta\to 1+\delta \qquad \beta'\to 1+\delta'.$$

Next we expand all the (gamma) factors which contain δ and δ' and the F_2 functions in Taylor series around $\delta=0$ and $\delta'=0$. In the end we take the limit of vanishing δ and δ'. This result is valid, as in the first case, within the region of convergence of the series which defines the F_2 function [13, 6],

$$F_2(\alpha,\beta,\beta';\gamma,\gamma'|z_1,z_2)=\sum_{j,k=0}^{\infty}\frac{z_1^j z_2^k}{j!k!}\frac{(\alpha|j+k)(\beta|j)(\beta'|k)}{(\gamma|j)(\gamma'|k)} \qquad |z_1|+|z_2|<1.$$

Following these steps we arrive at Davydychev's second expression for the Feynman integral [7]. Note that now the Mandelstam's variables s and t can never be zero. Moreover, as with the case of J_1, he has shown that the resulting expression for J_2 can be converted into the standard result in terms of sum of logarithms and dilogarithms through the use of a double integral representation for F_2 [13].

4. New results from NDIM

Using NDIM we have evaluated a massive box diagram integral, namely, a Feynman integral bearing four massive propagators. This integral is the one appearing in the photon–photon scattering in QED and the two well known results, expressed in terms of hypergeometric functions, have been easily found. So, the computation of such an integral, done as a 'lab test' for NDIM, has revealed a powerful technique, which transfers the intricacies of performing Feynman integrals in positive dimensions to that of solving a system of linear algebraic equations in negative dimensions, a far simpler task to perform than, for example, solving parametric integrals. Surprisingly, the technique not only reproduces the standard results, but also gives solutions covering other regions of the external momenta. The 10 remaining solutions for the system are as follows:

$$I_6 = f_6 \mathcal{S}_1\left(\alpha_6, \alpha'_6, \beta_6, \beta'_6, \theta_6; \gamma_6, \theta'_6 \left| \frac{-t}{s}, \frac{4m^2}{s}\right.\right) \tag{19}$$

$$I_7 = I_6(i \leftrightarrow k, j \leftrightarrow l | s \leftrightarrow t) \tag{20}$$

$$I_8 = f_8 \mathcal{S}_2\left(\alpha_8, \beta_8, \gamma_8, \delta_8, \phi_8; \rho_8, \phi'_8 \left| \frac{-t}{s}, \frac{-4m^2}{t}\right.\right) \tag{21}$$

$$I_9 = I_8(k \leftrightarrow l) \tag{22}$$

$$I_{10} = f_{10} \mathcal{S}_2\left(\alpha_{10}, \beta_{10}, \gamma_{10}, \delta_{10}, \phi_{10}; \rho_{10}, \phi'_{10} \left| \frac{-s}{t}, \frac{-4m^2}{s}\right.\right) \tag{23}$$

$$I_{11} = I_{10}(i \leftrightarrow j) \tag{24}$$

$$I_{12} = f_{12} \mathcal{S}_2\left(\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12}, \phi_{12}; \rho_{12}, \phi'_{12} \left| \frac{4m^2}{s}, \frac{-t}{4m^2}\right.\right) \tag{25}$$

$$I_{13} = I_{12}(k \leftrightarrow l) \tag{26}$$

$$I_{14} = f_{14} \mathcal{S}_2\left(\alpha_{14}, \beta_{14}, \gamma_{14}, \delta_{14}, \phi_{14}; \rho_{14}, \phi'_{14} \left| \frac{4m^2}{t}, \frac{-s}{4m^2}\right.\right) \tag{27}$$

$$I_{15} = I_{14}(i \leftrightarrow j) \tag{28}$$

where we have defined the following two functions

$$\mathcal{S}_1(\alpha, \alpha', \beta, \beta', \theta; \gamma, \theta' | z_1, z_2) = \sum_{\mu,\nu=0}^{\infty} \frac{z_1^\mu z_2^\nu}{\mu! \nu!} \frac{(\alpha|\mu)(\alpha'|\nu)(\beta|\mu)(\beta'|\nu)}{(\gamma|\mu+\nu)} \frac{(\theta|\mu+\nu)}{(\theta'|\mu+\nu)} \tag{29}$$

and

$$\mathcal{S}_2(\alpha, \beta, \gamma, \delta, \phi; \rho, \phi' | z_1, z_2) = \sum_{\mu,\nu=0}^{\infty} \frac{z_1^\mu z_2^\nu}{\mu! \nu!} \frac{(\alpha|\mu-\nu)(\beta|\mu)(\gamma|\nu)(\delta|\nu)}{(\rho|\mu)} \frac{(\phi|\nu-\mu)}{(\phi'|\nu-\mu)} \tag{30}$$

with

$$f_6 = \pi^{D/2} \left(\frac{s}{4}\right)^\sigma \frac{(-i|\sigma)(-j|\sigma)}{(\frac{1}{2} - \frac{1}{2}\sigma|\sigma + \frac{1}{4}D)(-\frac{1}{2}\sigma|\sigma + \frac{1}{4}D)}$$

$$f_8 = \pi^{D/2} s^l t^{\sigma - l} \frac{(-k|-i-j-\frac{1}{2}D)(-i|i-k+l)(-j|\sigma-l)}{(-i-l+\sigma|i+l+\frac{1}{2}D)}$$

$$f_{10} = \pi^{D/2} s^j t^{\sigma - j} \frac{(-l|-i-j-\frac{1}{2}D)(-i|i+k-l)(-j|\sigma-k)}{(-i-k+\sigma|i+k+\frac{1}{2}D)}$$

$$f_{12} = \pi^{D/2} s^l (-m^2)^{\sigma - l} \frac{(-k|l)(\sigma+\frac{1}{2}D|l-2\sigma-\frac{1}{2}D)}{(\sigma+\frac{1}{2}D|2l-2\sigma)}$$

$$f_{14} = \pi^{D/2} s^j (-m^2)^{\sigma - j} \frac{(-i|j)(\sigma+\frac{1}{2}D|j-2\sigma-\frac{1}{2}D)}{(\sigma+\frac{1}{2}D|2j-2\sigma)}$$

with the following parameters for the function $\mathcal{S}_1$:

$$\begin{aligned}
\alpha_6 &= -k \\
\alpha_6' &= \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D \\
\beta_6 &= -l \\
\beta_6' &= 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D \\
\theta_6 &= -\sigma \\
\gamma_6 &= 1 + i - \sigma \\
\theta_6' &= 1 + j - \sigma
\end{aligned}$$

and the parameters for the function $\mathcal{S}_2$:

$$\begin{aligned}
\alpha_8 &= j + k + \tfrac{1}{2}D & \alpha_{10} &= i + k + \tfrac{1}{2}D \\
\beta_8 &= -l & \beta_{10} &= -j \\
\gamma_8 &= 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D & \gamma_{10} &= 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D \\
\delta_8 &= \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D & \delta_{10} &= \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D \\
\phi_8 &= -i - j - k - \tfrac{1}{2}D & \phi_{10} &= -i - k - l - \tfrac{1}{2}D \\
\rho_8 &= 1 + k - l & \rho_{10} &= 1 + i - j \\
\phi_8' &= 1 - i - k - \tfrac{1}{2}D & \phi_{10}' &= 1 - i - l - \tfrac{1}{2}D
\end{aligned}$$

$$\begin{aligned}
\alpha_{12} &= \tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k - \tfrac{1}{2}l & \alpha_{14} &= \tfrac{1}{2} + \tfrac{1}{2}i - \tfrac{1}{2}j + \tfrac{1}{2}k + \tfrac{1}{2}l \\
\beta_{12} &= -l & \beta_{14} &= -j \\
\gamma_{12} &= -i & \gamma_{14} &= -k \\
\delta_{12} &= -j & \delta_{14} &= -l \\
\phi_{12} &= -i - j - k - \tfrac{1}{2}D & \phi_{14} &= -i - k - l - \tfrac{1}{2}D \\
\rho_{12} &= 1 + k - l & \rho_{14} &= 1 + i - j \\
\phi_{12}' &= -\tfrac{1}{2}i - \tfrac{1}{2}j - \tfrac{1}{2}k + \tfrac{1}{2}l & \phi_{14}' &= -\tfrac{1}{2}i + \tfrac{1}{2}j - \tfrac{1}{2}k - \tfrac{1}{2}l.
\end{aligned}$$

Observe that when the parameters θ and θ' in $\mathcal{S}_1$ are equal, then our defined function $\mathcal{S}_1$ becomes the known Appel's hypergeometric function F_3, whereas when the parameters ϕ and ϕ' in $\mathcal{S}_2$ are equal, our defined function $\mathcal{S}_2$ reduces to the known Appel's hypergeometric function H_2 (see section 5).

Looking carefully at these one can verify without difficulty that there are symmetry relations among them. For example, if we make the substitution $s \leftrightarrow t$, $i \leftrightarrow k$ and $j \leftrightarrow l$ in (19) we obtain (20). In a similar manner, the substitution $k \leftrightarrow l$ in (21) transforms it into (22) and the substitution $s \leftrightarrow t$, $j \leftrightarrow k$, $i \leftrightarrow l$ yields (22) $\leftrightarrow$ (23). There are several other symmetry properties of the box diagram which transform one solution into another.

Now we must combine them so as to have sums of linearly independent solutions bearing the same functional variable. This is the constructive prescription [4]. We then get from the above list six types of functional variables, that is, six new such combinations or six new results for the Feynman integral (5), namely,

$$J_3 = I_6 \qquad J_4 = I_7 \tag{31}$$

$$J_5 = I_8 + I_9 \qquad J_6 = I_{10} + I_{11} \tag{32}$$

$$J_7 = I_{12} + I_{13} \qquad J_8 = I_{14} + I_{15}. \tag{33}$$

Note that the relevant Feynman integral is obtained via

$$K(i, j, k, l; m) \equiv J(-i, -j, -k, -l; m).$$

5. Regularization and discussion

Here we constrain ourselves to the special case where the integral (5) is the one for QED photon–photon scattering at the one-loop level, that is, we are interested in taking the particular values $i = j = k = l = -1$. However, for some of the resulting expressions for the Feynman integral these values cannot be implemented straight away, because in the intermediate steps of the calculation they may become singular. Therefore some kind of regularization procedure is called for, but, of course, the final result is independent of the regularization procedure adopted (see the discussion following equation (18)). This is a regularization whereby the exponents of the propagators are modified (analytic regularization). Another difficulty to bear in mind is that Feynman integrals may be divergent for the physically meaningful dimension $D = 4$. The nature of this last singularity is completely different from the former. In all the results we have, the singularities arising from the exponents $i = j = k = l = -1$ cancel out. However, the poles in D *may* remain in the final expressions for the integral we are studying, because we are not considering the whole physical process of photon–photon scattering but only one of the integrals in it. Of course, if one considers the whole process, with all the diagrams in it, these poles also cancel out.

For the first two, i.e. J_3 and J_4, we can adopt the standard procedure of dimensional regularization [1]. Introduce $D = 4 - \varepsilon$ and expand the whole expression around $\varepsilon = 0$ to get

$$I_6 = \frac{8\pi^2}{s^2}\left[\frac{-2}{\varepsilon} + \log(-2\pi s) + \gamma_E\right] F_3(1, \tfrac{1}{2} + \tfrac{1}{2}\varepsilon, 1, 1 + \tfrac{1}{2}\varepsilon; 2 + \tfrac{1}{2}\varepsilon | x, y) \tag{34}$$

where $x = -t/s$, $y = 4m^2/s$, F_3 is a hypergeometric function of two variables which is absolutely convergent for $|x| < 1$ and $|y| < 1$, and γ_E is Euler's constant [6, 13]. We can write a simpler expression by using a reduction formula [6, 13, 18],

$$F_3(\alpha, \alpha', \beta, \gamma - \beta; \gamma | x, y) = \frac{1}{(1-y)^{\alpha'}} F_1(\beta, \alpha, \alpha'; \gamma | x, z) \tag{35}$$

where $z = y/(y-1)$ and F_1 is another hypergeometric function of two variables which is absolutely convergent in the same region of the F_3 above. This function has a simple integral representation [13],

$$F_1(\alpha, \beta, \beta'; \gamma | z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \mathrm{d}u \, \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-uz_1)^{\beta}(1-uz_2)^{\beta'}} \tag{36}$$

where the parameters must satisfy $\mathrm{Re}\,(\alpha) > 0$ and $\mathrm{Re}\,(\gamma - \alpha) > 0$. It is straightforward to evaluate this integral when the parameters take the values we are using. Substituting (36) and (35) in (34) and expanding the hypergeometric function in Taylor series, we get

$$J_3(-1,-1,-1,-1;m) \equiv I_6 = \frac{8\pi^2}{s(s-4m^2)}\left[\frac{-2}{\varepsilon} - \partial_{\beta'} - \partial_\gamma + \log(-2\pi s) + \gamma_E + \log\left(1 - \frac{4m^2}{s}\right)\right] F_1(\alpha, \beta, \beta'; \gamma | x, z). \tag{37}$$

Note that there is a simple pole which we did not expect by simple power counting. We will discuss this singularity and the one that appears in the following solution in the next section. Here we introduce the parametric derivatives [7, 15],

$$\frac{\partial(\alpha|z)}{\partial\alpha} \equiv \partial_\alpha(\alpha|z) = (\alpha|z)[\psi(\alpha+z) - \psi(\alpha)] \tag{38}$$

where the ψ-function is the logarithmic derivative of the gamma function [13, 17]. First carry out the parametric derivatives in (36), then substitute the values of the parameters and integrate. For the other terms the integral results in

$$F_1(1, 1, \tfrac{1}{2}; 2|x, z) = -\frac{s}{t}\frac{1}{R_{st}} \log\left(\frac{1+R_{st}}{1-R_{st}}\frac{R_s - R_{st}}{R_s + R_{st}}\right) \tag{39}$$

where

$$R_s = \sqrt{1 - \frac{4m^2}{s}} \qquad R_{st} = \sqrt{1 - \frac{4m^2}{t} - \frac{4m^2}{s}}. \tag{40}$$

We can write down immediately the result for the integral I_7 by noting that it can be transformed into I_6 if we make the changes $i \leftrightarrow k$, $j \leftrightarrow l$ and $s \leftrightarrow t$,

$$J_4(-1,-1,-1,-1;m) \equiv I_7 = \frac{8\pi^2}{t(t-4m^2)}\left[\frac{-2}{\varepsilon} - \partial_{\beta'} - \partial_\gamma + \log(-2\pi t) + \gamma_E + \log\left(1 - \frac{4m^2}{t}\right)\right] F_1(\alpha, \beta, \beta'; \gamma | w, w') \tag{41}$$

where $w = -s/t$ and $w' = 4m^2/(4m^2 - t)$. For the region of convergence see figure 2.

For the remaining solutions dimensional regularization is unsuitable to regularize their divergences. Consider, for example, (21) where there is a factor $(-i|i-k+l)$ which is divergent in the particular limit we are interested in, i.e. $i = j = k = l = -1$. This factor

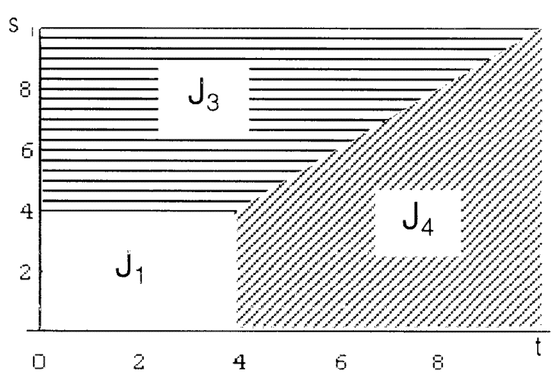

Figure 2. Region of absolute convergence of the solutions J_1, J_3 and J_4.

has no D-dependence and dimensional regularization here is useless. What we must do is to use a different procedure, namely, regularizing the exponent of some of the propagators [7, 16].

Let us then consider the fifth solution of the Feynman integral, J_5. We must regularize one exponent of one of the propagators, say, $k = -1 - \zeta$ (we could also take the exponent l). The important point is that the final result will be independent of this choice. The other exponents are set to minus one while the dimension of the spacetime remains arbitrary. Doing this we have

$$I_8 = \pi^{D/2} \frac{1}{st^{3-D/2}} \frac{\Gamma(3-\frac{1}{2}D+\zeta)\Gamma(\zeta)\Gamma^2(\frac{1}{2}D-2-\zeta)}{t^{\zeta}\Gamma(1+\zeta)\Gamma(D-4-\zeta)} \times H_2\left(-\zeta, 1, 1+\frac{1}{2}\zeta, \frac{1}{2}+\frac{1}{2}\zeta, 1-\zeta \,\middle|\, \frac{-t}{s}, \frac{-4m^2}{t}\right) \tag{42}$$

and

$$I_9 = \pi^{D/2} \frac{\Gamma(3-\frac{1}{2}D)\Gamma^2(\frac{1}{2}D-2)}{st^{3-D/2}} \frac{\Gamma(-\zeta)}{s^{\zeta}\Gamma(D-4-\zeta)} \times H_2\left(0, 1+\zeta, 1+\frac{1}{2}\zeta, \frac{1}{2}+\frac{1}{2}\zeta, 1+\zeta \,\middle|\, \frac{-t}{s}, \frac{-4m^2}{t}\right). \tag{43}$$

The hypergeometric function H_2 is defined by the double sum [13],

$$H_2(\alpha, \beta, \gamma, \delta; \rho|x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha|m-n)(\beta|m)(\gamma|n)(\delta|n)}{(\rho|m)} \frac{x^m y^n}{m!n!}. \tag{44}$$

The region of absolute convergence of this function $H_2(\ldots|x, y)$ is bounded by the lines [13],

$$|y| < \frac{1}{1+|x|} \qquad |x| < 1 \qquad |y| < 1 \tag{45}$$

see figure 3.

Proceeding with our analysis of the new results, for the solution J_5, let us now expand the H_2 function in Taylor series around $\zeta = 0$, keeping terms up to the first order in ζ

$$J_5(-1,-1,-1,-1; m) = \frac{2^{5-D}\pi^{D/2}\sqrt{\pi}\,\Gamma(3-\frac{1}{2}D)\Gamma(\frac{1}{2}D-2)}{st^{3-D/2}\Gamma(\frac{1}{2}D-\frac{3}{2})}$$

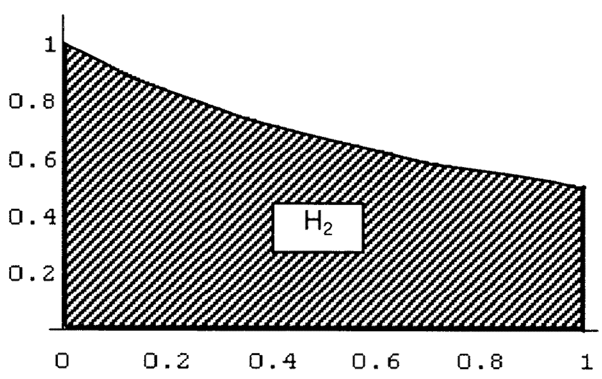

Figure 3. Region of absolute convergence of the H_2 hypergeometric function.

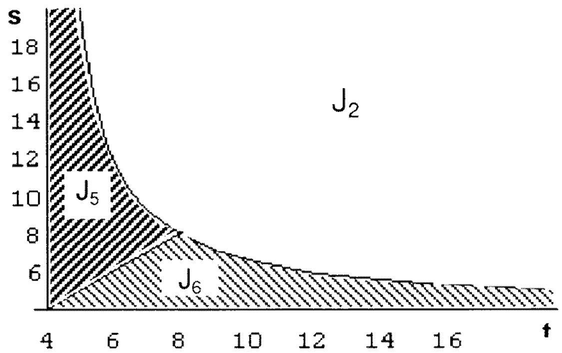

Figure 4. Region of absolute convergence of J_2, J_5 and J_6. The curve that separates J_2 of the others is a branch cut, see equation (63).

$$\times\left[-\gamma_E+\log\left(\frac{s}{t}\right)-2\psi\left(\frac{1}{2}D-1\right)+\psi\left(3-\frac{1}{2}D\right)+\frac{4}{D-4}-\partial_\alpha-\partial_\rho\right]$$
$$\times H_2\left(\alpha,1,1,\frac{1}{2};\rho\left|\frac{-t}{s},\frac{-4m^2}{t}\right.\right) \tag{46}$$

where the parametric derivatives must be taken at the point $\alpha=0$; $\rho=1$. The nature and meaning of these singularities will be the touched on in the following section. *Observe here that only the divergence in $D=4$ remains. The apparent divergences in $\zeta=0$ have cancelled out*.

In a similar manner we regularize the sixth solution. But now we take $i=-1-\zeta$. As a result we get

$$J_6(-1,-1,-1,-1;m)=\frac{2^{5-D}\pi^{D/2}\sqrt{\pi}\Gamma(3-\frac{1}{2}D)\Gamma(\frac{1}{2}D-2)}{ts^{3-D/2}\Gamma(\frac{1}{2}D-\frac{3}{2})}$$
$$\times\left[-\gamma_E+\log\left(\frac{t}{s}\right)-2\psi\left(\frac{1}{2}D-1\right)+\psi\left(3-\frac{1}{2}D\right)+\frac{4}{D-4}-\partial_\alpha-\partial_\rho\right]$$
$$\times H_2\left(\alpha,1,1,\frac{1}{2};\rho\left|\frac{-s}{t},\frac{-4m^2}{s}\right.\right) \tag{47}$$

and as we shall see later on, these two functions H_2 are related to the functions F_3 that are divergent also. The region of convergence can be constructed as we did above for the H_2 function (see figure 4).

Consider now the seventh solution of the Feynman integral, J_7. Like the preceding case, it has a simple pole in the exponents, so that it demands only one suitable parameter to regularize it. Looking at (25) and (26) we note that a good choice to introduce our regularization parameter is to take $l=-1-\zeta$, while the other exponents can be set to $i=j=k=-1$ without any problem. Then,

$$I_{12}=\frac{-\pi^2}{m^2s}\left(-\frac{1}{\zeta}-1+\log s+\mathrm{O}(\zeta)\right)$$
$$\times\mathcal{S}_2\left(-\frac{1}{2}+\frac{1}{2}\zeta,1+\zeta,1,1,3-\frac{1}{2}D;1+\zeta,1-\frac{1}{2}\zeta\left|\frac{4m^2}{s},\frac{-t}{4m^2}\right.\right) \tag{48}$$

and

$$I_{13} = \frac{-\pi^2}{m^2 s}\left(\frac{1}{\zeta} - 1 - \gamma_E - \log(-m^2) + \mathrm{O}(\zeta)\right)$$
$$\times \mathcal{S}_2\left(-\frac{1}{2} - \frac{1}{2}\zeta, 1, 1, 1, 3 - \frac{1}{2}D + \zeta; 1 - \zeta, 1 + \frac{1}{2}\zeta \middle| \frac{4m^2}{s}, \frac{-t}{4m^2}\right). \tag{49}$$

Now expand the factors of (48), (49) and the series (30) around $\zeta = 0$ and substitute the values $\alpha = -\frac{1}{2}$, $\beta = \gamma = \delta = \rho = \phi = \phi' = 1$. Using the fact that $\partial_\beta + \partial_\rho = 0$ (only because these two parameters are equal) and an analogous relation between ϕ and ϕ', we get, in four dimensions,

$$J_7(-1, -1, -1, -1; m) = \frac{\pi^2}{m^2 s}\left[2 + \gamma_E + \partial_\alpha + \partial_\rho - \log\left(\frac{-s}{m^2}\right)\right]$$
$$\times H_2\left(\alpha, 1, 1, 1; \rho \middle| \frac{4m^2}{s}, \frac{-t}{4m^2}\right). \tag{50}$$

Note that the above result is finite and that there is no dependence on ϕ and ϕ', so $\mathcal{S}_2$ reduces to H_2. The pole cancels out and then we can take the limit of vanishing ζ.

The next two solutions follow the same procedure, yielding

$$J_8(-1, -1, -1, -1; m) = \frac{\pi^2}{m^2 t}\left[2 + \gamma_E + \partial_\alpha + \partial_\rho - \log\left(\frac{-t}{m^2}\right)\right]$$
$$\times H_2\left(\alpha, 1, 1, 1; \rho \middle| \frac{4m^2}{t}, \frac{-s}{4m^2}\right) \tag{51}$$

which is also finite. The region of convergence of this solution is shown in figure 5. The parametric derivatives have already been calculated by Davydychev [7]. Using the transformation formula between H_2 and F_2 [18],

$$H_2(\alpha, \beta, \gamma, \delta; \rho | x, y) = \mathcal{A}_1 F_2\left(\alpha + \gamma, \beta, \gamma; \rho, 1 + \gamma - \delta \middle| x, \frac{-1}{y}\right)$$
$$+ \mathcal{A}_2 F_2\left(\alpha + \delta, \beta, \delta; \rho, 1 + \delta - \gamma \middle| x, \frac{-1}{y}\right) \tag{52}$$

where we define the coefficients

$$\mathcal{A}_1 = \frac{\Gamma(1-\alpha)\Gamma(\delta-\gamma)}{\Gamma(\delta)\Gamma(1-\alpha-\gamma)} y^{-\gamma} \qquad \mathcal{A}_2 = \mathcal{A}_1(\gamma \leftrightarrow \delta) \tag{53}$$

we can identify the parametric derivatives of H_2 with the ones of F_2 calculated by Davydychev. Care must be taken with (52) because with the particular parameters we are using the individual terms on the RHS are singular, but their singularities cancel out when both terms are added together.

5.1. Discussion

As we have mentioned earlier, the set of new solutions we have obtained here contains singular solutions that deserve a closer look. Let us examine them in order to understand the meaning and the nature of such singularities. To begin with, we give some arguments to show the correctness of our results.

Consider the first result,

$$J_1(-1, -1, -1, -1; m) = \frac{\pi^2}{6m^4} F_3\left(1, 1, 1, 1; \frac{5}{2} \middle| \frac{s}{4m^2}, \frac{t}{4m^2}\right). \tag{54}$$

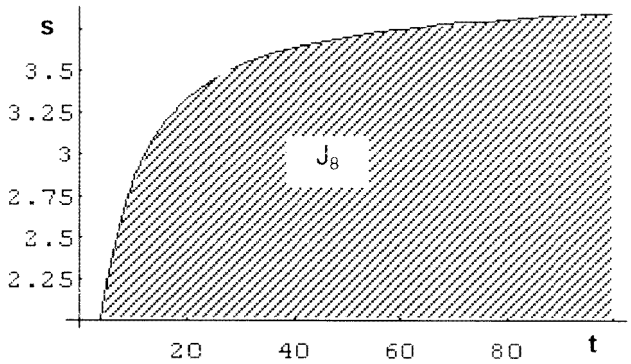

Figure 5. Region of absolute convergence of the J_8. The solution J_7 is symmetric to it in $s \leftrightarrow t$. They are finite and hold in the relativistic regime of forward scattering in the t and s-channel respectively.

The hypergeometric function F_3 which appears here is related to the hypergeometric function H_2 via analytic continuation (see Erdélyi [18]),

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma | x, y) = \mathcal{B}_1 H_2 \left(1 + \alpha - \gamma, \alpha, \alpha', \beta'; 1 + \alpha - \beta \middle| \frac{1}{x}, -y \right)$$
$$+ \mathcal{B}_2 H_2 \left(1 + \beta - \gamma, \beta, \alpha', \beta'; 1 + \beta - \alpha \middle| \frac{1}{x}, -y \right) \qquad (55)$$

where the two coefficients are

$$\mathcal{B}_1 = \frac{\Gamma(\beta - \alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-x)^{-\alpha} \qquad \mathcal{B}_2 = \mathcal{B}_1(\alpha \leftrightarrow \beta). \qquad (56)$$

So, with the help of equation (55) we rewrite F_3 in terms of H_2 without worrying very much about constant factors because they arrange themselves properly in the process. Indeed, in this case both factors on the RHS containing gamma functions are singular (this is a special case of analytic continuation known as the logarithmic case), but whose singularities cancel out, leaving us with a finite result as it should be. Then,

$$J_1 \sim F_3 \left(1, 1, 1, 1; \frac{5}{2} \middle| \frac{s}{4m^2}, \frac{t}{4m^2} \right) = C_1 H_2 \left(-\frac{1}{2}, 1, 1, 1; 1 \middle| \frac{4m^2}{s}, \frac{-t}{4m^2} \right)$$
$$+ C_2 H_2 \left(-\frac{1}{2}, 1, 1, 1; 1 \middle| \frac{4m^2}{s}, \frac{-t}{4m^2} \right) \qquad (57)$$

which clearly portrays the same H_2 function we have in J_7. Conclusion: NDIM provides, even if we did not know (55) *a priori*, the transformation $J_1 \to J_7$, or, in other words, the analytic continuation formula $F_3 \to H_2$. Moreover, as Erdérlyi [18] mentioned, there is a transformation similar to (55) for the variable y in F_3. This will give $J_1 \to J_8$.

Next, in order to verify that there are branch points in the Feynman integral, we can do the following. Consider the definition of the hypergeometric function H_2 given in (44). Substituting the values of the parameters—recall that the derivatives of an analytic function are also analytic having the same region of convergence—two of them cancel out and we get

$$H_2 \left(-\frac{1}{2}, 1, 1, 1; 1 \middle| x, y \right) = \sum_{\mu, \nu = 0}^{\infty} \frac{(1|\nu)(1|\nu)}{(\frac{1}{2}|\nu)} \frac{(-y)^\nu}{\nu!} \left(-\frac{1}{2} - \nu \middle| \mu \right) \frac{x^\mu}{\mu!} \qquad (58)$$

where we have used the identity $(a|-k) = (-1)^k/(1-a|k)$. Observe that the series in μ is a hypergeometric function ${}_1F_0$ [17] that can be summed. It results in the following

$$H_2\left(-\frac{1}{2}, 1, 1, 1; 1 \middle| x, y\right) = \sqrt{1-x}\sum_{\nu=0}^{\infty}\frac{(1|\nu)(1|\nu)}{(\frac{1}{2}|\nu)}\frac{[-y(1-x)]^\nu}{\nu!} \qquad (59)$$

with variables x and y given in either J_7 or J_8. The remaining series in ν is a ${}_2F_1$ hypergeometric function that can be written down in terms of an elementary function (an arcsin one) and it is straightforward to show that it has branch points, see (63) below.

The same procedure can be applied to J_3. Using (55) for the hypergeometric function F_3, we get

$$J_3 \sim F_3\left(1, 1, 1, \frac{1}{2}; 2 \middle| \frac{-t}{s}, \frac{4m^2}{s}\right) = C_3 H_2\left(0, 1, 1, \frac{1}{2}; 1 \middle| \frac{-s}{t}, \frac{-4m^2}{s}\right) + C_4 H_2\left(0, 1, 1, \frac{1}{2}; 1 \middle| \frac{-s}{t}, \frac{-4m^2}{s}\right) \qquad (60)$$

yielding $J_3 \to J_6$. The analogous transformation for the variable y in F_3 yields $J_3 \to J_5$ and so on.

As above, it is possible to express this H_2 function in terms of an elementary function, this time a square root. Considering its definition, the cancelling of the parameters β and ρ for the specified values and summing the series in ν we get,

$$H_2\left(0, 1, 1, \frac{1}{2}; 1 \middle| x, y\right) = \sum_{\nu=0}^{\infty}\frac{[-y(1-x)]^\nu(\frac{1}{2}|\nu)}{\nu!} = \frac{1}{\sqrt{1+y(1-x)}}. \qquad (61)$$

Observe that the square root in the denominator is equal to R_{st}, see equation (40). An important point to note is that even though the results remain divergent, they are still connected by an analytic continuation formula. The questions that need to be then addressed are: *What does this mean? What is the nature of these singularities?*

First of all, it is known [12] that a four-point graph like the one in the photon–photon scattering has no leading singularities in the physical region. Such singularities do happen to occur in four-point functions when the two incoming particles enter the same vertex and the two outgoing particles also leave the same vertex (see figure 6). However, since this is not our case, we conclude that the singularities we have do not occur on the physical sheet [12].

Secondly, in analytically continuing a given function from a region $\mathcal{R}_1$ into another region, $\mathcal{R}_2$ it is important that no singularities be present between the regions, otherwise the result for the analytic continuation may not be unique. The non-uniqueness always manifests itself whenever the singularity is of the branch-point type [19]. We know that for the photon–photon scattering process we have a branch-cut in $s = 4m^2$ in the s-channel, so that in carrying out our analytic continuation from $J_6 \to J_3$, we are crossing this branch-cut, and then the singularities do arise.

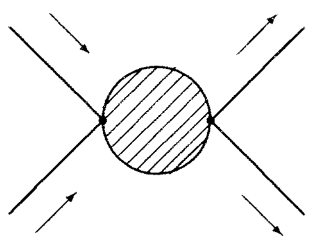

Figure 6. The general 2-particle $\to$ 2-particle Feynman graph that has leading singularity(ies) on the physical sheet [12].

This simple logic shows us the great possibilities of NDIM. It reproduces three general—with no restriction in the parameters—analytic continuation relations between Appel's hypergeometric functions which are far from trivial to obtain (see [18]). The technique also allows us the added bonus of pinpointing singularities of Feynman integrals.

Eden [20] devised a technique to find out the singularities of integral representations. In [12] Eden *et al* applied it in the general box diagram and the equation of Landau's surface—the surface of possible singularities of a integral representation—is given by a 4×4 determinant. In our case (equal mass for the virtual matter fields and on-shell photons) the Landau's equation [1, 12, 21] is,

$$\frac{st}{4m^6}\left(\frac{st}{4m^2}-s-t\right)=0 \tag{62}$$

so that there are four possible solutions,

$$s=0 \qquad t=0 \qquad s=\frac{4m^2t}{t-4m^2} \qquad t=\frac{4m^2s}{s-4m^2}. \tag{63}$$

It is important to observe that the two last solutions make the hypergeometric function H_2 in (61) and in (59) singular. *They are branch points of the Feynman integral*. The first two are the so-called pseudo-threshold [1, 12, 21]—singularities of the Feynman integral which occur on an unphysical sheet—see also that the possible singularities of (5) are located in the region of convergence of the two above functions, J_3 and J_4. This is why the poles do not cancel: because of the so-called pinch singularities. We can verify, comparing the analysis contained in [1], that the last two solutions of Landau's equation are in fact singularities of (5).

6. Conclusion

Using NDIM we have evaluated a massive box diagram integral, namely, a Feynman integral bearing four massive propagators. This scalar integral is the one appearing in the photon–photon scattering in QED and the two well known results, expressed in terms of hypergeometric functions, have been easily found. So, the computation of such an integral, done as a 'lab test' for NDIM, has revealed a powerful technique, which transfers the intricacies of performing Feynman integrals in positive dimensions to that of solving a system of linear algebraic equations in negative dimensions, a far simpler task to perform than, for example, solving parametric integrals. More than that, surprisingly, the technique not only reproduces the standard results, but gives simultaneously, solutions covering other regions of the external momenta.

Acknowledgments

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